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A Numerical computation for solving the fractional delay differential equations via the MQ-RBF collocation method

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Abstract

In the paper, a numerical technique based on the multiquadric radial basis function (MQ-RBF) collocation method is presented for the fractional delay differential equations. The convergence analysis and the unique fixed point of the collocation methods are propounded in the L_2 norm. Numerical results illustrate the credibility and applicability of the numerical technique. The obtained results are compared with those obtained by other methods.

1. Introduction

Nowadays, fractional calculus is a name for the theory of integrals and derivatives of arbitrary order that called fractional integrals and derivatives [1]. On the other hand, this topic has been a particular interest during the past three decades. Fractional calculus arises in many scientific disciplines such as the mathematical systems, electro-dynamics, viscoelasticity, bade analysis of feedback amplifiers, capacitor theory, electrical circuits, chemistry, biology, control theory and fitting of experimental data [3-5]. Various applications can be elegantly modeled with the help of the fractional derivatives and integrals. For example, the numerical solution of Multi-order fractional differential equations considered by Kai Diethelm [6]. In [7] the convergence of the variational iteration method for solving multi-delay differential equations is given, and the convergence of the VIM for solving linear systems of ODEs with constant coefficients has been discussed in [8]. The most methods have been used to solve fractional partial differential equations, fractional differential equations, fractional backward differential, fractional integro-differential equations and dynamic systems with multi-order methods and Adomian's decomposition methods in during the last decades [9, 10]. In [11] investigated the collocation method based on Multiquadric radial basis functions (MQ-RBF) for fractional differential equations. Authors used the collocation method, to solve the time fractional diffusion equation and showed convergence order of the time discrete scheme [27]. Heryudono is directed towards localized RBF approximation of unity collocation method for convection-diffusion equations proposed in [13]. We know, the idea of delay plays an important role in fractional differential equations. The researchers are considering various methods for solving the fractional delay differential equations. Some of these methods are as: Wang et al. used the Adams-Bashforth-Multon method and the Grï¼nwald-Letnikov definition to approximate the solution of the delay fractional order differential equations [14, 15]. Rad discussed the operational matrix of the Bernstein's and Legendre polynomials to solve the fractional delay differential equations with Tau method in [16]. Recently, Saedshoear et al. discussed fractional backward differential formulas for fractional delay differential equations with periodic and anti-periodic conditions [17]. Also, Saedshoear in [18] investigates the FBDF5 method for delay differential equations of fractional order with periodic and anti-periodic conditions.

In this paper, the main intention is to introduce the collocation RBFs method to solve the fractional delay differential equations (FDDEs) of the form

$$\begin{aligned}
&\sum_{i=0}^{n-1} \mathfrak{h}_{n-i}\,{}_0^c\mathbb{D}_t^{\beta_{n-i}}y(t) + \mathfrak{h}_{n+1}y(t-\zeta) = f(t) \quad t \in [0,T],\\
&y(t) = g(t), \quad t \in [-\zeta, 0],\\
&\mathfrak{h}_{n-i} \in \mathbb{R}, \quad i = 0,1,2,\ldots,n-1, \quad \mathfrak{h}_{n+1} \neq 0,\\
&0 \leqslant \beta_1 < \beta_2 <\ldots< \beta_n < 1, \quad T > 0,
\end{aligned} \tag{1}$$

with $y(0) = -y(T)$, anti-periodic condition or $y(0) = y(T)$ periodic condition where ${}_0^c\mathbb{D}_t^{\beta}$ is the Caputo derivative.

The outline of this paper is as follows: In the first section, we explain some basic compliment and mathematical preliminaries of fractional derivatives. The second section discusses radial basic functions and interpolation for our subsequent expansion. In the third Section, we summarize the collocation methods of the basic MQ-RBFs for the numerical solution of fractional delay differential equations. Convergence analysis and the unique fixed point of the collocation methods based on the basis MQ-RBFs are presented in the fourth section. The last section with the numerical instance and results demonstrate the accuracy of the scheme.

2 Preliminaries and Notations of Fractional Derivatives

In this section, we reminisce some definitions and properties of fractional derivatives and integrals which will be used later. There are different definitions, which do not coincide in general. This epitome regards two of them, namely, the Riemann-Liouville and the Caputo fractional operator [1, 4].

Corollary 1 *Now, let us consider an observation of Cauchy's formula for repeated integration. Suppose $g(x)$ be a Lebesgue integrable function defined on the interval $[a,b]$ i.e. $g(x) \in L^1(a,b)$. The integral operator $\mathfrak{I}$, for $x \in [a,b]$ is defined*

$$\mathfrak{I}g(x) = \int_a^x g(x)dx.$$

Cauchy demonstrated that repeated of this integral operator can be expressed with a single integral

$$\begin{aligned}
\mathfrak{I}^n g(x) &= \int_a^x \int_a^{\tau_n} \ldots \int_a^{\tau_2} g(\tau_1)d\tau_1 \ldots d\tau_{n-1}d\tau_n\\
&= \frac{1}{(n-1)!}\int_a^x (x-\tau)^{n-1}g(\tau)d\tau,
\end{aligned}$$

holds for $n \in \mathbb{N}, a, \tau \in \mathbb{R}, \tau > a$. If a positive real number β is replaced with n, a formula for fractional integration is obtained [1].

Definition 1 *The Riemann-Liouville fractional integral of order β is*

$$\mathfrak{I}^\beta g(x) = \frac{1}{\Gamma(\beta)}\int_a^x (x-\tau)^{\beta-1}g(\tau)d\tau.$$

Suppose that $\beta > 0, x > a, \beta, a, x \in \mathbb{R}$ [1].

Remark 1 *The Property of integral operator is the identity operator i.e. $\mathfrak{I}^0 g(x) = g(x)$. Another property is the linearity*

$$\mathfrak{I}^\beta(\gamma f(x) + \eta g(x)) = \gamma\mathfrak{I}^\beta f(x) + \eta\mathfrak{I}^\beta g(x), \quad \beta \in \mathbb{R}^+, \gamma, \delta \in \mathbb{C},$$

Suppose $g(x)$ is continuous for $x \geq 0$, the following equalities hold

$$\begin{aligned}
\lim_{\beta \to 0}\mathfrak{I}^\beta g(x) &= g(x),\\
\mathfrak{I}^\beta(\mathfrak{I}^\alpha g(x)) &= \mathfrak{I}^\alpha(\mathfrak{I}^\beta g(x)) = \mathfrak{I}^{\beta+\alpha}g(x), \quad \beta, \alpha \in \mathbb{R}^+, \gamma, \delta \in \mathbb{C}.
\end{aligned}$$

Definition 2 *The Riemann-Liouville derivative or the Riemann-Liouville operator of fractional order $\beta > 0$, for a function $g\colon (0,\infty) \longrightarrow \mathbb{R}$ is defined as*

$$ {}_{0}^{RL}\mathbb{D}_t^{\beta} g(t) = \begin{cases} \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \int_0^t \frac{g(\tau)}{(t-\tau)^{\beta-n+1}} d\tau, & n-1<\beta<n, \\ \frac{d^n}{dt^n} g(t), & \beta = n, \end{cases} \tag{2} $$

we note that $n-1 \leq \beta < n$, $n \in \mathbb{N}$ and $t \in \mathbb{R}$ [1].

Definition 3 *The Caputo derivative or the Caputo operator of fractional order $\beta > 0$ is defined as [19]*

$$ {}_{0}^{C}\mathbb{D}_t^{\beta} g(t) = \begin{cases} \frac{1}{\Gamma(n-\beta)} \int_0^t \frac{g^{(n)}(\tau)}{(t-\tau)^{\beta-n+1}} d\tau, & n-1<\beta<n, \\ \frac{d^n}{dt^n} g(t), & \beta = n. \end{cases} \tag{3} $$

We note that $n-1 \leq \beta < n$ and $n \in \mathbb{N}$. This derivative is defined by the Italian mathematician Caputo in 1967 [2].

Definition 4 *The Riemann-Liouville fractional integral of order β is*

$$ \Im^{\beta} g(x) = \frac{1}{\Gamma(\beta)} \int_a^x (x-\tau)^{\beta-1} g(\tau) d\tau. $$

Suppose that $\beta > 0, x > a, \beta, a, x \in \mathbb{R}$ [1].

Remark 2 *The Property of integral operator is the identity operator i.e. $\Im^0 g(x) = g(x)$. Another property is the linearity*

$$ \Im^{\beta}(\gamma f(x) + \eta g(x)) = \gamma \Im^{\beta} f(x) + \eta \Im^{\beta} g(x), \quad \beta \in \mathbb{R}^+, \gamma, \delta \in \mathbb{C}, $$

Suppose $g(x)$ is continuous for $x \geq 0$, the following equalities hold

$$ \begin{aligned} \lim_{\beta \to 0} \Im^{\beta} g(x) &= g(x), \\ \Im^{\beta}(\Im^{\alpha} g(x)) &= \Im^{\alpha}(\Im^{\beta} g(x)) = \Im^{\beta+\alpha} g(x), \quad \beta, \alpha \in \mathbb{R}^+, \gamma, \delta \in \mathbb{C}. \end{aligned} $$

Remark 3 *As it is already known from the previous the Riemann-Liouville 2 and the Caputo fractional derivatives 3 do not coincide. Unlike the Riemann-Liouville case, for which*

$$ \lim_{\beta \to n-1} {}_{0}^{RL}\mathbb{D}_t^{\beta} g(t) = g^{(n-1)}(x), $$

for the Caputo case

$$ \lim_{\beta \to n-1} {}_{0}^{C}\mathbb{D}_t^{\beta} g(t) = g^{(n-1)}(x) - g^{(n-1)}(0), $$

as it is proved in [1].

3 Radial Basis Functions

Radial basis functions (RBFs) are widely used for interpolation and approximation of ordinary differential equations and PDEs [20]. In this section, we introduce radial basis functions. Suppose $\boldsymbol{\sigma}_m^d$ be the subspace of $C(\mathbb{R}^d)$ including of all d-variate algebraic polynomials of degree less than m.

Assume $\upsilon_1, \upsilon_2, \ldots, \upsilon_N, \in \mathbb{R}^d$ are distinct points corresponding scalar function values $y(\upsilon_1), \ldots, y(\upsilon_N)$. The standard approximation is of the form

$$ \mathfrak{S}\mathbb{Y}(x) = \sum_{i=1}^{N} c_i \phi_i(x - \upsilon_i) + \sum_{j=1}^{\mathrm{I}} \gamma_j P_j(x), \tag{4} $$

where $\mathrm{I} = dim \quad \boldsymbol{\sigma}_m^d$, $c_i, \gamma_j \in \mathbb{R}$ and $\{P_j\}_{j=1}^{\mathrm{I}}$ is a basis of $\boldsymbol{\sigma}_m^d$. $\{\phi_i\}_{i=1}^{N}$ are the real-valued functions of order m. The most well–known radial basis functions are [20-22, 26, 26-33]

$$ \begin{aligned} &\phi(x) = (-1)^{k+1} x^{2k} log(x), k \in \mathbb{N}, \ (Thin\ plate\ splines) \\ &\phi(x) = (-1)^{\lceil \frac{\beta}{2} \rceil} (\lambda^2 + x^2)^{\frac{\beta}{2}}, \beta \geq 0, \beta \in 2\mathbb{N}, (Multi\ quadrics (MQ)) \\ &\phi(x) = (\lambda^2 + x^2)^{\frac{\beta}{2}}, \beta \in 2\mathbb{Z}, \lambda \neq 0, (Inverse\ multi\ quadrics) \\ &\phi(x) = exp(\frac{-x^2}{\lambda^2}), \lambda > 0, (Gaussians). \end{aligned} \tag{5} $$

The parameter λ is announced a shape parameter and manages the flatness of the RBFs. Also, the notation $[\frac{\beta}{2}]$ indicates the smallest integer greater than $\frac{\beta}{2}$. The shape parameter has a considerable effect on the precision of the RBF interpolation. In this article, we apply multiquadric that is almost easiest and choose λ based on trial.

4 MQ-RBF Collocation Scheme

In this section, our target is to portray the MQ-RBF collocation method for the numerical solution of FDDEs. For this work in equation (1), we assume $T = \zeta$, otherwise, we modify the Eq. (1) with a linear transform. According to the Caputo fractional derivative (3), the equation (1) takes the following form [17, 21, 22]

$$
\begin{aligned}
&\sum_{i=0}^{n-1} \frac{\mathfrak{h}_{n-i}}{\Gamma(1-\beta_{n-i})} \int_0^t \frac{y'(\tau)}{(t-\tau)^{\beta_{n-i}}} d\tau + \mathfrak{h}_{n+1} y(t-\zeta) = f(t), \quad t \in [0,T],\\
&y(t) = g(t), \quad t \in [-\zeta, 0],\\
&\mathfrak{h}_{n-i} \in \mathbb{R}, \quad i = 0,1,2,\dots,n-1, \quad \mathfrak{h}_{n+1} \neq 0,\\
&0 \leqslant \beta_1 < \beta_2 < \dots < \beta_n < 1, \quad 0 \leq \beta_{n-i} \leq 1, \quad T > 0.
\end{aligned} \tag{6}
$$

The approximation of the function $y'(x)$ may be compounded as a linear combination of MQ-RBF basis functions. We usually take the following form:

$$= \sum_{i=1}^{N} c_i (-1)^{[\frac{\alpha}{2}]} (\lambda^2 + (x - x_i)^2)^{\frac{\alpha}{2}} + \sum_{j=0}^{m-1} \gamma_j x^j.$$

We suppose $\alpha = 1$. Also, If $\mathfrak{S}\mathbb{Y}(x)$ be the approximation of $y'(x)$ and $\psi_i(x) = (-1)^{[\frac{\alpha}{2}]}(\lambda^2 + (x - x_i)^2)^{\frac{\alpha}{2}}$. Therefore we have

$$\mathfrak{S}\mathbb{Y}(x) = \sum_{i=1}^{N} c_i \psi_i(x) + \sum_{j=0}^{m-1} \gamma_j x^j, \tag{7}$$

where $\{c_i\}_{i=1}^{N}$ and $\{\gamma_j\}_{j=0}^{m-1}$ are unknown coefficients and λ is shape parameter. In this method, we chose the shape parameter by random. By substituting Eq. (7) into Eq. (6) and $T = \zeta$, we have:

$$
\begin{aligned}
&\sum_{i=0}^{n-1} \Gamma_{n-i} \int_0^t \frac{\mathfrak{S}\mathbb{Y}(\tau)}{(t-\tau)^{\beta_{n-i}}} d\tau + \mathfrak{h}_{n+1} g(t) = f(t) \quad t \in [0,T],\\
&\Gamma_{n-i} = \frac{\mathfrak{h}_{n-i}}{\Gamma(1-\beta_{n-i})},
\end{aligned} \tag{8}
$$

with summarizing the above equation, we have

$$
\begin{aligned}
&\sum_{i=0}^{n-1} \sum_{j=1}^{N} \Gamma_{n-i} . c_j \int_0^t \frac{\psi_j(\tau)}{(t-\tau)^{\beta_{n-i}}} d\tau + \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \Gamma_{n-i} . \gamma_j \int_0^t \frac{\tau^j}{(t-\tau)^{\beta_{n-i}}} d\tau = \mathbb{F}(t),\\
&\mathbb{F}(t) = f(t) - \mathfrak{h}_{n+1} g(t), \quad t \in [0, \zeta].
\end{aligned} \tag{9}
$$

Therefore, to find the function $y(t)$ by MQ-RBFs, we have $N + m$ unknown coefficients. To determine these unknown coefficients, we exert the equation 9 at a set of collocation points $\boldsymbol{\chi} = \{x_j\}_{j=1}^{N} \subseteq [0, \zeta]$. Note to determine the unknown $\{\gamma_j\}$ $j = 0,1,\dots,m-1$, we use m conditions as

$$\sum_{i=1}^{N} c_i (x_i)^j = 0, \quad 0 \leq j \leq m-1. \tag{10}$$

Then, we yield a linear system with $N + m$ linear equations in $N + m$ unknowns as

$$
\begin{cases}
\sum_{i=0}^{n-1} \sum_{j=1}^{N} c_j \int_0^{x_k} \Phi_{i,j}(x_k, \tau) d\tau + \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \gamma_j \int_0^{x_k} P_{i,j}(x_k, \tau) d\tau = \mathbb{F}(x_k),\\
1 \leq k \leq N\\
\sum_{i=1}^{N} c_i (x_i)^j = 0, \quad 0 \leq j \leq m-1
\end{cases} \tag{11}
$$

where $\Phi_{i,j}(x_k, \tau) = \Gamma_{n-i} \frac{\psi_j(\tau)}{(x_k-\tau)^{\beta_{n-i}}}$ and $P_{i,j}(x_k, \tau) = \Gamma_{n-i} \frac{\tau^j}{(x_k-\tau)^{\beta_{n-i}}}$. Solving the equation 11 gives us the unknown coefficients $\{c_i\}_{i=1}^{N}$ and $\{\gamma_j\}_{j=0}^{m-1}$. Because the above integral solves analytically in system 11, we did not use numerical methods for approximating the integral solution. If this integral was not solved numerically, we would use Legendre-Gauss-Lobatto method, which is precise for the $2m - 1$ degree.

5 The convergence analysis

In this section, a unique numerical method and upper error boundary Will be reviewed for the fractional delay differential equations (1). To perform so, we represent some lemmas and theorems.

Definition 5 *Shifted surface splines are defined as*

$$\phi(x)=\begin{cases}(-1)^{\lceil m-\frac{d}{2}\rceil}(x^2+a^2)^{m-\frac{d}{2}}, & d \ is \ odd,\\ (-1)^{m-\frac{d}{2}+1}(x^2+a^2)^{m-\frac{d}{2}}\log(x^2+a^2)^{\frac{1}{2}}, & d \ is \ even,\end{cases} \tag{12}$$

where $d,m\in\mathbb{N}, m>\frac{d}{2}$ and $[m-\frac{d}{2}]$ indicates the smallest integer greater than $m-\frac{d}{2}$. These functions include all of the multiquadric, power and thin plate splines [24].

Lemma 1 *If $\mathfrak{S}y(x)$ in 4 be the interpolant to $y(x)$ using shifted surface spline ϕ in 12, then we have the following relationship for every function $y\in W_2^m$*

$$|y-\mathfrak{S}y|_{m,L_2(\Omega)}\leq|y-\mathfrak{S}y|_{\phi}\leq|y|_{\phi}, \tag{13}$$

where $y(x)$ be a function in the space $\mathfrak{F}_{\phi}$, and ϕ is strictly conditionally positive definite of order m. Also, Ω is an open domain in $\mathbb{R}^d$ [24].

Note the space $\mathfrak{F}_{\phi}$ is the real vector space consisting of all functions $f\in C(\mathbb{R}^d)$ and W_2^m is the special case of the Sobolev spaces W_p^m that is called Hilbert space.

Theorem 1 *Suppose that the parameter λ in the basis function ϕ is chosen to be proportionate to h. Assume $\mathfrak{S}y(x)$ be the approximation of $y(x)$ using the shifted surface spline ϕ. Then we have an upper error bound of the form*

$$||y-\mathfrak{S}y||_{L_p(\Omega)}\leq Ch^{\gamma_p}|y|_{m,L_2(\mathbb{R}^d)},\quad 1\leq p\leq\infty, \tag{14}$$

where $h=sup_{x\in\Omega}min_{x_j\in X}|x-x_j|$ that X is the density in Ω. Also, C is a positive constant and γ_p equal to $min(m,m-d/2+d/p)$ [24].

Definition 6 *(Fixed Point Property). A mapping $\mathbb{T}$ on a metric space (X,d) into itself is said to have a fixed point y if there exists $y\in X$ such that $\mathbb{T}y=y$ [25].*

Theorem 2 *(Banach Contraction Mapping). Suppose $\mathbb{T}:X\longrightarrow X$ be a contraction mapping with contractivity factor α on a complete metric space (X,d) into itself. Then the mapping $\mathbb{T}$ possesses exactly one fixed point $y\in X$. Moreover, the sequence $\{\mathbb{T}^n(x)\}_{n=0}^{n=k}, x\in X$ converges to the point y: that is $lim_{n\to\infty}\mathbb{T}^n(x)=y$ [25].*

Theorem 3 *Suppose $y(x)\in W_\infty^m(\Omega)\cap\ \mathfrak{F}_{\phi}$ be the exact solution of (1), then the equation (1) has a unique solution.*

Proof. The equation (1) tantamount the following formulation

$${}_0^cD_t^{\beta_n}y(t)=\frac{1}{y_n}f(t)-\frac{y_{n+1}}{y_n}y(t-\xi)-\sum_{i=1}^{n-1}h_{n-i}{}_0^cD_t^{\beta_{n-i}}y(t),\qquad t\in[0,\zeta].$$

Therefore

$${}_0^c\mathbb{D}_t^{\beta_n}y(t)=\mathfrak{y}(t)-\mathfrak{A}y(t), \tag{15}$$

where $\mathfrak{y}(t)=\frac{1}{\mathfrak{h}_n}f(t)-\frac{\mathfrak{h}_{n+1}}{\mathfrak{h}_n}g(t)$ and $\mathfrak{A}y(t)=\sum_{i=1}^{n-1}\mathfrak{h}_{n-i}{}_0^c\mathbb{D}_t^{\beta_{n-i}}y(t)$. By taking the integral operator ${}_0\mathfrak{I}_t^{\beta_n}$ of equation 15 and suppose

$$\mathbb{T}y=y(t)=y(T)+\frac{1}{\Gamma(\beta_n)}\int_0^t(t-\tau)^{\beta_n-1}(\mathfrak{y}(\tau)-\mathfrak{A}y(\tau))d\tau. \tag{16}$$

If $\tilde{y}(x)$ be a solution of (1), then $\tilde{y}(x)$ is applied in 16. For applying (2), suppose X be the set of all real-valued continuous functions on $[0,1]$. X is a closed subset of the metric space $C[0,1]$ with sup metric, hence a complete metric space and propound the map

$$\mathbb{T}: C[0,1] \longrightarrow C[0,1], \quad 1.5cm\mathbb{T}y = h, \tag{17}$$

where $h(t) = y(T) + \frac{1}{\Gamma(\beta_n)}\int_0^t (t-\tau)^{\beta_n - 1}(\mathfrak{y}(\tau) - \mathfrak{A}y(\tau))d\tau$. Since $h(t) \in X$ so $\mathbb{T}$ is well defined. Now we prove that $\mathbb{T}$ is a contraction mapping of X into itself.

$$\begin{aligned} d(\mathbb{T}y, \mathbb{T}\tilde{y}) &= \frac{1}{\Gamma(\beta_n)} sup \left|\int_0^t (t-\tau)^{\beta_n-1}\mathfrak{A}(y(\tau) - \tilde{y}(\tau))d\tau\right| \\ &\leq \frac{1}{\Gamma(\beta_n)} sup \int_0^t \left|(t-\tau)^{\beta_n-1}\right| |\mathfrak{A}(y(\tau) - \tilde{y}(\tau))| d\tau. \end{aligned} \tag{18}$$

Because the function $\left|(t-\tau)^{\beta_n-1}\right|$ is continuous and positive in the interval $[0,t] \subseteq [0,1]$ and the function $|\mathfrak{A}(y(\tau) - \tilde{y}(\tau))|$ is also continuous. So $t \in [0,t]$ exist that

$$\begin{aligned} d(\mathbb{T}y, \mathbb{T}\tilde{y}) &\leq \frac{1}{\Gamma(\beta_n)} sup|\mathfrak{A}(y(t) - \tilde{y}(t))| \int_0^t \left|(t-\tau)^{\beta_n-1}\right| d\tau \\ &\leq \frac{(\tau)^{\beta_n}}{\Gamma(1+\beta_n)} sup|\mathfrak{A}(y(t) - \tilde{y}(t))| \\ &\leq \frac{(n-1)\times(\tau)^{\beta_n}}{\Gamma(1+\beta_n)} sup\left|{}_0^c\mathbb{D}_t^\gamma(y(t) - \tilde{y}(t))\right|. \end{aligned} \tag{19}$$

In the above equation, it suffices to prove that equation $\left|{}_0^c\mathbb{D}_t^\gamma(y(t) - \tilde{y}(t))\right|$ applies to the Lipschitz condition. For this work, we prove that $\left|\frac{d}{dt}\right|_0^c \mathbb{D}_t^\gamma(y(t) - \tilde{y}(t))|$ is bounded. Since the Lipschitz condition of the continuous equation $\left|\mathbb{D}_t^\gamma(y(t) - \tilde{y}(t))\right|$ is equivalent to having the boundedness $\left|\frac{d}{dt}\mathbb{D}_t^\gamma(y(t) - \tilde{y}(t))\right|$. Since $y'(0) - \tilde{y}'(0) = 0$ So

$$\begin{aligned} \left|\frac{d}{dt}\mathbb{D}_t^\gamma(y(t) - \tilde{y}(t))\right| &= \left|\mathbb{D}_t^{\gamma+1}(y(t) - \tilde{y}(t))\right| \\ &= \frac{1}{\Gamma(1+\gamma)}\left|\int_0^t \frac{y''(t) - \tilde{y}''(t)}{(t-\tau)^\gamma} d\tau\right| \\ &\leq \frac{1}{\Gamma(1+\gamma)}\int_0^t |y''(t) - \tilde{y}''(t)||(t-\tau)^{-\gamma}| d\tau \\ &\leq \frac{(\tau)^{1-\gamma}}{(1-\gamma)\times\Gamma(1+\gamma)} |y''(\varepsilon) - \tilde{y}''(\varepsilon)| \\ &\leq \frac{(\tau)^{1-\gamma}}{(1-\gamma)\times\Gamma(1+\gamma)} \times Max|y''(\varepsilon) - \tilde{y}''(\varepsilon)|. \end{aligned} \tag{20}$$

As a result, $P \in \mathbb{R}^+$ exists.

$$\left|\mathbb{D}_t^\gamma(y(t) - \tilde{y}(t))\right| \leq |P(y(t) - \tilde{y}(t))|. \tag{21}$$

From 19 and 21 we conclude that

$$d(\mathbb{T}y, \mathbb{T}\tilde{y}) \leq P \times \frac{(n-1)\times(\tau)^{\beta_n}}{\Gamma(1+\beta_n)} d(y, \tilde{y}). \tag{22}$$

We conclude with $0 \leq P \times \frac{(n-1)\times(\tau)^{\beta_n}}{\Gamma(1+\beta_n)} < 1$ that Eq. (15) has a unique fixed point.

Theorem 4 *[25] Suppose $\tilde{y}(x)$ be the approximation solution of $y(x) \in W_\infty^m(\Omega) \cap \mathfrak{F}_\phi$ using MQ-RBF basis functions 7, then we have an error bound of the form*

$$||y(x) - \tilde{y}(x)||_{L_2(\Omega)} \leq C|\Gamma|\sqrt{\frac{(\tau)^{2\beta_n-1}}{2\beta_n-1}} \times \frac{M\times(1+\beta)\times Ch^m}{\Gamma(1-\beta)} |y|_{m,L_2(\Omega)}, \tag{23}$$

where $|\Gamma| = Max\Gamma(1-\beta_n)$.

6 Numerical procedures

To support our numerical discussion, we represent the efficiency, veracity and precision of the proposed method. In these experiments, all of the calculation is implemented in using the Mathematica software and the numerical results are compared with paper [17], which is solved by the fractional backward differential formulas (FBDF).

Example 1 *Consider the following FDDE with an anti-periodic condition*

$$ {}_0^C\mathbb{D}_t^{0.2}y(t)+y(t-1)=\frac{\Gamma(3)t^{1.8}}{\Gamma(2.8)}-\frac{\Gamma(2)t^{0.8}}{\Gamma(1.8)}+t^2-3t+1, \qquad t\in[0,2] $$
$$ \mathrm{y(t)}=t^2-t-1, \quad t\in[-1,0] $$
$$ y(0)=-y(2). $$

The exact solution is $y(t)=t^2-t-1$ [17]. In this article, we solve this problem with the proposed method and reported the results for different points (N) and shape parameters of RBF (λ) at final time $T=\zeta=2$. The Tables 1 and 2 represent the absolute error of the $y(1)$ and max absolute error in [0,1] with using different shape parameters. Also, Table 3 compares absolute error in paper [17] and our method. MEVS is the maximum error of various shape parameters.

Table 1: Approximate errors for Example 1 using $\lambda=1.5,2$

	$\lambda=1.5$		$\lambda=2$	
n	*Max*[0,1]	*y*(1)	*Max*[0,1]	*y*(1)
3	5.37438×10^{-4}	5.30194×10^{-4}	3.13809×10^{-4}	3.11457×10^{-4}
5	1.48520×10^{-5}	1.43985×10^{-5}	5.19376×10^{-6}	5.07595×10^{-6}
10	3.17777×10^{-8}	3.17777×10^{-8}	8.05070×10^{-9}	6.98492×10^{-10}
15	1.87855×10^{-8}	1.13187×10^{-8}	4.88344×10^{-9}	5.92991×10^{-10}
20	2.04783×10^{-8}	6.51562×10^{-9}	4.87977×10^{-9}	4.87977×10^{-10}
100	2.17346×10^{-8}	1.55226×10^{-8}	3.43599×10^{-9}	2.50430×10^{-10}
200	2.14955×10^{-8}	2.14955×10^{-8}	3.21862×10^{-9}	2.29829×10^{-10}

Table 2: Approximate errors for Example 1 using $\lambda=2.5,3$

	$\lambda=2.5$		$\lambda=3$	
n	*Max*[0,1]	*y*(1)	*Max*[0,1]	*y*(1)
3	2.03561×10^{-4}	2.04533×10^{-4}	1.43492×10^{-4}	1.43021×10^{-4}
5	2.23465×10^{-6}	2.19196×10^{-6}	1.10790×10^{-6}	1.08872×10^{-6}
10	2.97681×10^{-8}	2.02413×10^{-8}	1.04936×10^{-8}	7.19592×10^{-9}
15	3.45980×10^{-8}	1.02659×10^{-8}	1.30291×10^{-8}	3.24917×10^{-9}
20	4.10122×10^{-8}	3.01770×10^{-9}	1.48460×10^{-8}	6.80302×10^{-10}
100	3.82086×10^{-8}	3.42079×10^{-8}	1.20954×10^{-8}	1.35401×10^{-8}
200	4.21683×10^{-8}	4.21683×10^{-8}	1.49707×10^{-8}	1.49707×10^{-8}

Table 3: Comparison of errors in Example 1 with [17]

n	h	*MEVS*	*FBDF4* [17]
10	0.1	3.17777×10^{-8}	$1.7381765302294\times10^{-2}$
20	0.05	6.51562×10^{-9}	$1.0250950209419\times10^{-2}$
100	0.01	3.42079×10^{-8}	$2.8756916022030\times10^{-3}$
200	0.005	4.21683×10^{-8}	$1.6550266584620\times10^{-3}$

Example 2 *Consider the following FDDE with a periodic condition*

$$ {}_0^C\mathbb{D}_t^{0.4}y(t)+{}_0^C\mathbb{D}_t^{0.3}y(t)+y(t-1)=\frac{\Gamma(3)t^{1.6}}{\Gamma(2.6)}-\frac{\Gamma(2)t^{0.6}}{\Gamma(1.6)}+\frac{\Gamma(3)t^{1.7}}{\Gamma(2.7)}-\frac{\Gamma(2)t^{0.7}}{\Gamma(1.7)}+t^2-3t+2, \ \ t\in[0,1] $$
$$ \mathrm{y(t)}=t^2-t, \quad t\in[-1,0] $$
$$ y(0)=y(1). $$

The exact solution is $y(t)=t^2-t$ [17]. We solve this problem with the proposed method and reported the results for various points (N) and shape parameter of RBF (λ) at final time $T=\zeta=1$. The Tables 4 and 5 represent the absolute error of the $y(1)$ and max absolute error in $[0,1]$ with using different shape parameters. Also, Table 6 compares the absolute error in paper[17] and our method.

Table 4: Approximate errors for Example 2 using $\lambda = 1.5,2$

	$\lambda = 1.5$		$\lambda = 2$	
n	$Max[0,1]$	$y(1)$	$Max[0,1]$	$y(1)$
3	1.19948×10^{-3}	9.66752×10^{-4}	7.04126×10^{-4}	5.70251×10^{-4}
5	3.58196×10^{-5}	2.55446×10^{-5}	1.26605×10^{-5}	9.06387×10^{-6}
10	1.97068×10^{-8}	5.06407×10^{-9}	1.43371×10^{-8}	7.36327×10^{-9}
15	1.04732×10^{-8}	4.19095×10^{-9}	1.96138×10^{-8}	5.68434×10^{-13}
20	3.66708×10^{-9}	3.66708×10^{-9}	3.13157×10^{-8}	8.84757×10^{-9}
100	3.11084×10^{-9}	6.03905×10^{-10}	6.97569×10^{-9}	2.91766×10^{-9}
200	3.01626×10^{-9}	1.24055×10^{-9}	6.54641×10^{-9}	3.73484×10^{-9}

Table 5: Approximate errors for Example 2 using $\lambda = 2.5,3$

	$\lambda = 2.5$		$\lambda = 3$	
n	$Max[0,1]$	$y(1)$	$Max[0,1]$	$y(1)$
3	4.60219×10^{-4}	3.73491×10^{-4}	3.23393×10^{-4}	2.62731×10^{-4}
5	5.47820×10^{-6}	3.92717×10^{-6}	2.72479×10^{-6}	1.95441×10^{-6}
10	2.15694×10^{-8}	2.14204×10^{-8}	5.7459×10^{-7}	1.93715×10^{-7}
15	4.45638×10^{-9}	3.60887×10^{-9}	1.48979×10^{-8}	6.05362×10^{-9}
20	1.58325×10^{-8}	1.58325×10^{-8}	1.86065×10^{-7}	1.45288×10^{-7}
100	3.8380×10^{-8}	2.12835×10^{-8}	1.35927×10^{-8}	7.53789×10^{-9}
200	3.77164×10^{-8}	2.83630×10^{-8}	1.00522×10^{-8}	1.33445×10^{-8}

Table 6: Comparison of errors in Example 2 with [17]

n	h	*MEVS*	*FBDF4 [17]*
10	0.1	1.93715×10^{-7}	$1.559229520712 \times 10^{-3}$
20	0.05	1.45288×10^{-7}	$5.10934619817 \times 10^{-4}$
100	0.01	2.12835×10^{-8}	$3.67922074200 \times 10^{-5}$
200	0.005	2.83630×10^{-8}	$1.18622713050 \times 10^{-5}$

Table 7: Relative errors for Example 2 and Example 1 using $\lambda = 2$

n	*Example 1*	*Example 2*
3	2.22632×10^{-4}	2.88597×10^{-3}
5	2.89429×10^{-6}	4.09612×10^{-5}
10	2.57465×10^{-9}	6.43788×10^{-8}
15	1.69734×10^{-9}	2.22754×10^{-8}
20	1.53676×10^{-9}	1.56755×10^{-8}
100	1.15223×10^{-9}	1.44647×10^{-8}
200	1.10940×10^{-9}	1.34745×10^{-8}

7 Conclusions

The MQ-RBF collocation method is used to solve the fractional delay differential equations on $[0,1]$. Some properties of fractional calculations are presented and the collocation approximation is utilized to reduce the solution of FDDEs to the solution of a linear system of equations with the sparse matrix of coefficients. Compared to other methods, using the RBF led to a more effective and faster algorithm. We proved the unique fixed point and investigated convergence analysis of the collocation methods. The convergence behavior and a unique solution of the methods are corroborated by some numerical instances with the periodic and anti-periodic condition. The numerical solutions will be invaluable since the analytical solution for this problems is very intricate. The numerical results have been compared with other existing methods and performed better in every case. Through Table 7, we can also see that the relative errors for $\lambda = 2$ with increasing the n, the error is monotonically decreasing. The main advantage of this method is its simplicity and small computation costs.

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